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LINEAR PRESERVERS OF ROW-DENSE MATRICES

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*Dedicated to the memory of Professor Miroslav Fiedler,
who originated the idea of dense matrices.*

Abstract. Let $\mathbf{M}_{m,n}$ be the set of all $m \times n$ real matrices. A matrix $A \in \mathbf{M}_{m,n}$ is said to be row-dense if there are no zeros between two nonzero entries for every row of this matrix. We find the structure of linear functions $T\colon \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ that preserve or strongly preserve row-dense matrices, i.e., $T(A)$ is row-dense whenever A is row-dense or $T(A)$ is row-dense if and only if A is row-dense, respectively. Similarly, a matrix $A \in \mathbf{M}_{n,m}$ is called a column-dense matrix if every column of A is a column-dense vector. At the end, the structure of linear preservers (strong linear preservers) of column-dense matrices is found.

Keywords: row-dense matrix; linear preserver; strong linear preserver

MSC 2010: 15A04, 15A21

1. Introduction

The study of linear preserver problems is one of the active research topics in matrix theory and linear algebra which concerns the characterization of linear maps on spaces of matrices that preserve certain special properties, or special subsets. Some kinds of linear preserver problems were presented in [4], [5] and [6]. Let $\mathbf{M}_{m,n}$ be the set of all $m \times n$ real matrices, and as usual $\mathbb{R}_n = \mathbf{M}_{1,n}$ and $\mathbf{M}_n = \mathbf{M}_{n,n}$. One of the oldest known linear preserver problems is due to Frobenius. In 1897 he found the structure of linear maps $T\colon \mathbf{M}_n \to \mathbf{M}_n$ that preserve determinant, i.e, $\det(T(A)) = \det(A)$ for all $A \in \mathbf{M}_n$, see [2]. Some recent works on linear preserver problems can be found in [3] and [7].

The notion of dense matrices originated in [1]. A vector $a \in \mathbb{R}_n$ is called a *row-dense vector* if there are no zeros between two nonzero components of a. A matrix $A \in \mathbf{M}_{m,n}$ is called a *row-dense matrix* if every row of A is a row-dense vector.

Definition 1.1. Let $T\colon \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ be a linear map. We say that T preserves or strongly preserves row-dense matrices, if $T(A)$ is row-dense whenever A is row-dense or $T(A)$ is row-dense if and only if A is row-dense, respectively.

In this paper we first classify the linear maps $T\colon \mathbb{R}_n \to \mathbb{R}_n$ that preserve (strongly preserve) row-dense vectors and then the linear maps $T\colon \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ that preserve (strongly preserve) row-dense matrices are characterized. Since for $n \leqslant 2$ every $A \in \mathbf{M}_{m,n}$ is a row-dense matrix and consequently every linear map on $\mathbf{M}_{m,n}$ is a linear preserver of row-dense matrices, throughout the paper we assume that $n \geqslant 3$.

2. Linear preservers of row-dense vectors

In this section we show some properties of linear functionals $f\colon \mathbb{R}_n \to \mathbb{R}$. Also we characterize all linear maps $T\colon \mathbb{R}_n \to \mathbb{R}_n$ that preserve (strongly preserve) row-dense vectors. The following lemmas are used to obtain the main results.

Lemma 2.1. *Let $g, f_1, \ldots, f_k\colon \mathbb{R}_n \to \mathbb{R}$ be nonzero linear functionals such that g is not a multiple of any of the coordinate axes projections. Then*

(i) $\{(x_1, \ldots, x_n)\colon x_i \neq 0, \forall\, i\} \cap \mathrm{Ker}(g)$ *is a dense subset of* $\mathrm{Ker}(g)$.

(ii) *If there exists $a \in \mathrm{Ker}(g)$ such that $f_1(a), \ldots, f_k(a)$ are nonzero, then there exists $b \in \mathrm{Ker}(g)$ with no zero component such that $f_1(b), \ldots, f_k(b)$ are nonzero.*

P r o o f. We first prove (i). For every $a = (a_1, \ldots, a_n) \in \mathrm{Ker}(g)$ and for every $\varepsilon > 0$ choose $0 < \delta < \min\{\varepsilon/n, |a_i|\colon 1 \leqslant i \leqslant n,\ a_i \neq 0\}$. There exists a basis $\mathcal{B}$ of $\mathrm{Ker}(g)$ contained in $\mathrm{Ker}(g) \cap B(a, \delta)$. Now, consider the following two cases:

Case 1. Let a have $n-1$ nonzero components. With the choice of δ for every element $b = (b_1, \ldots, b_n) \in \mathcal{B}$, if $a_i \neq 0$ then $b_i \neq 0$. So, all elements of $\mathcal{B}$ have at least $n-1$ nonzero components in the same positions. Since g is not a multiple of any of the coordinate axes projections, some of the elements of $\mathcal{B}$ have no zero entries. Hence, for every $\varepsilon > 0$, $\{(x_1, \ldots, x_n)\colon x_i \neq 0,\ \forall\, i\} \cap \mathrm{Ker}(g) \cap B(a, \varepsilon) \neq \emptyset$.

Case 2. Let $a = (a_1, \ldots, a_n)$ have k nonzero components where $0 \leqslant k \leqslant n-2$. Without loss of generality assume that $a_1, \ldots, a_k$ are nonzero and $a_{k+1} = \ldots = a_n = 0$. For every $b = (b_1, \ldots, b_n) \in \mathcal{B}$, with the choice of δ it is clear that $b_1, \ldots, b_k$ are nonzero. If $b_{k+1} = \ldots = b_n = 0$ for all $b = (b_1, \ldots, b_n) \in \mathcal{B}$, then $\dim(\mathrm{Ker}(g)) < n-1$. However, for any linear functional g, the dimension of $\mathrm{Ker}(g)$ equals n or $n-1$.

Hence, we have a contradiction. So some elements of $\mathcal{B}$ have at least $k+1$ nonzero components. Choose such an element, say a', and the corresponding $B(a', \delta')$. By continuing this process we reach Case 1.

Thus, $\{(x_1, \ldots, x_n)\colon x_i \neq 0, \forall i\} \cap \operatorname{Ker}(g)$ is a dense subset of $\operatorname{Ker}(g)$. This completes the proof of (i).

For the proof of (ii), define the function $f\colon \mathbb{R}_n \to \mathbb{R}$ by $f(x) = f_1(x) \ldots f_k(x)$. Since f is continuous and $f(a) \neq 0$, there exists $\delta > 0$ such that $f(x) \neq 0$ for all $x \in B(a, \delta)$. By part (i), there exists a vector $b \in \operatorname{Ker}(g) \cap B(a, \delta)$ with no zero component, as desired. $\square$

For a linear map $T\colon \mathbb{R}_n \to \mathbb{R}_m$, by $[T]$ we mean the $n \times m$ matrix representation of T with respect to the standard bases of $\mathbb{R}_n$ and $\mathbb{R}_m$, and hence $T(x) = x[T]$ for all $x \in \mathbb{R}_n$.

Lemma 2.2. *Let $T\colon \mathbb{R}_n \to \mathbb{R}_m$ be a linear map defined by*

$$T(x) = [\lambda_1 f(x), \ldots, \lambda_k f(x), \mu_1 g(x), \ldots, \mu_l g(x), \eta_1 h(x), \ldots, \eta_p h(x)],$$

where $f, g, h\colon \mathbb{R}_n \to \mathbb{R}$ are nonzero linear functionals such that $\{f, g\}$ and $\{g, h\}$ are linearly independent, and $\lambda_1, \ldots, \lambda_k$, $\mu_1, \ldots, \mu_l$, $\eta_1, \ldots, \eta_p$ are nonzero scalars with $k + l + p = m$. If T preserves row-dense vectors, then there exists $1 < j_0 < n$ such that $g(e_j) = 0$ for all $j \neq j_0$ and one of the following holds:

(i) *$f(e_j) = 0$ for all $j > j_0$ and $h(e_j) = 0$ for all $j < j_0$;*
(ii) *$f(e_j) = 0$ for all $j < j_0$ and $h(e_j) = 0$ for all $j > j_0$.*

In other words, there exist rank one matrices A, B and C such that

$$[T] = \begin{bmatrix} A & C & 0 \\ 0 & 0 & B \end{bmatrix} P,$$

where C has no zero entry in its last row and any preceding rows of C are zero rows, A and B have no zero columns, the last row of C and the first row of B lie in the same row of $[T]$, and P is the identity or the backward identity matrix of order m.

Proof. It is easy to see that if $\operatorname{Ker}(g) \subseteq \operatorname{Ker}(f) \cup \operatorname{Ker}(h)$, then $\operatorname{Ker}(g) \subseteq \operatorname{Ker}(f)$ or $\operatorname{Ker}(g) \subseteq \operatorname{Ker}(h)$. So $\{f, g\}$ or $\{g, h\}$ is linearly dependent, which is a contradiction. Hence, there exists $a \in \operatorname{Ker}(g)$ such that $f(a)$ and $h(a)$ are nonzero. Next, assume that g is not a multiple of any of the coordinate axes projections. Then by Lemma 2.1, there exists $b \in \operatorname{Ker}(g)$ with no zero component such that $f(b)$ and $h(b)$ are nonzero. Therefore, b is dense but $T(b)$ is not dense, which is a contradiction. So, there exists $1 \leqslant j_0 \leqslant n$ such that $g(e_j) = 0$ for all $j \neq j_0$ and $g(e_{j_0}) \neq 0$. Assume, if possible, that (i) and (ii) don't hold. Then there exist some $i, j > j_0$ or $i, j < j_0$

such that $f(e_i) \neq 0 \neq h(e_j)$. Without loss of generality assume that $i \leqslant j < j_0$ and choose $\alpha, \beta \in \mathbb{R}$ such that $f(a) \neq 0 \neq h(a)$ where $a = \alpha e_i + \beta e_j + \sum\limits_{k=i+1}^{j-1} e_k$. So $g(a) = 0$ and $f(a) \neq 0 \neq h(a)$ and hence, a is dense but $T(a)$ is not dense, which is a contradiction. Therefore, (i) or (ii) holds and the proof is complete. $\square$

Lemma 2.3. *Let $T\colon \mathbb{R}_n \to \mathbb{R}_n$ be a linear map with components $f_1, \ldots, f_n\colon \mathbb{R}_n \to \mathbb{R}$. Then*

(i) *If T preserves row-dense vectors and if for some $1 \leqslant i \leqslant j \leqslant n$, $f_i, f_j \neq 0$, then for all $i \leqslant k \leqslant j$, $f_k \neq 0$.*

(ii) *If for some $1 \leqslant i \leqslant n-1$, $f_i \neq 0$ and $\{f_i, f_{i+1}\}$ is linearly dependent, then the linear map $S\colon \mathbb{R}_n \to \mathbb{R}_{n-1}$ defined by $S = (f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n)$ preserves row-dense vectors if and only if T preserves row-dense vectors.*

Proof. To prove (i), without loss of generality, assume, if possible, that $f_1 \neq 0 = f_2 \neq f_3$. Then there exist $1 \leqslant i, j \leqslant n$ such that $f_1(e_i) \neq 0 \neq f_3(e_j)$. Choose $\alpha, \beta \in \mathbb{R}$ such that $f_1(a) \neq 0 \neq f_3(a)$ where $a = \alpha e_i + \beta e_j + \sum\limits_{k=i+1}^{j-1} e_k$ if $i \leqslant j$ and similarly if $j \leqslant i$. Therefore $T(a)$ is not dense which is a contradiction and the proof of (i) is complete. The proof of (ii) is easy to see. $\square$

In the following theorem we denote the j^{th} axes projection on $\mathbb{R}_n$ by π_j.

Theorem 2.4. *Let $T\colon \mathbb{R}_n \to \mathbb{R}_n$ be a linear map. Then T preserves row-dense vectors if and only if the $n \times n$ matrix $[T]$ is of one of the following forms:*

(i) *There exist rank one matrices A and B with nonzero columns such that*

$$[T] = [\,0 \quad A \quad B \quad 0\,], \tag{2.1}$$

where any of the block columns might be absent.

(ii) *There exist rank one matrices $C_1, \ldots, C_k$, A and B such that*

$$[T] = \left[\begin{array}{c|ccccc|c} & A & C_1 & & & 0 & \\ & & & C_2 & & & \\ 0 & & & & \ddots & & 0 \\ & & & & & C_k & \\ & 0 & & & & B & \end{array}\right] P, \tag{2.2}$$

where for $1 \leqslant j \leqslant k$ the last row of each C_j has no zero entry and any preceding rows of C_j are zero rows, A and B have no zero columns, the last row of C_k and the first row of B lie in the same row of $[T]$, P is the identity or the backward identity matrix of order n, and where any of the block columns might be absent.

P r o o f. First assume that $T \neq 0$ preserves row-dense vectors. Let the linear functionals $f_1, \ldots, f_n\colon \mathbb{R}_n \to \mathbb{R}$ be the components of T, so $T = (f_1, \ldots, f_n)$. By part (ii) of Lemma 2.3, by removing the additional components of T, we obtain the linear map $S\colon \mathbb{R}_n \to \mathbb{R}_m$ such that $\{g_i, g_{i+1}\}$ are linearly independent for all $1 \leqslant i \leqslant m-1$, where $S = (g_1, \ldots, g_m)$. If $m \leqslant 2$, it is easy to see that (i) holds. Assume that $m \geqslant 3$ and consider $\{g_1, g_2, g_3\}$. By Lemma 2.2, g_2 is a multiple of an axes projection, i.e. $g_2 = \lambda_2 \pi_{j_2}$ for some $1 \leqslant j_2 \leqslant n$, $\lambda_2 \in \mathbb{R}$, and one of the following holds:

(1) $g_1(e_j) = 0$ for all $j > j_2$ and $g_3(e_j) = 0$ for all $j < j_2$;
(2) $g_1(e_j) = 0$ for all $j < j_2$ and $g_3(e_j) = 0$ for all $j > j_2$.

We assume that (1) holds; the other case is similar. If $m = 3$, the proof is complete and if $m \geqslant 4$, consider $\{g_2, g_3, g_4\}$. Again by using Lemma 2.2, $g_3 = \lambda_3 \pi_{j_3}$ for some $1 \leqslant j_3 \leqslant n$, $\lambda_3 \in \mathbb{R}$ and one of the following holds:

(3) $g_2(e_j) = 0$ for all $j > j_3$ and $g_4(e_j) = 0$ for all $j < j_3$;
(4) $g_2(e_j) = 0$ for all $j < j_3$ and $g_4(e_j) = 0$ for all $j > j_3$.

Since $\{g_2, g_3\}$ is linearly independent, $j_2 \neq j_3$. If $j_3 < j_2$, then by (1), $g_3 = 0$ which is a contradiction and hence $j_2 < j_3$. If (4) holds, then $g_2 = 0$ which is a contradiction and hence (3) holds. By continuing this process $m-2$ times, we find $1 \leqslant j_2 < \ldots < j_{m-1} \leqslant n$ and $\lambda_2, \ldots, \lambda_{m-1} \in \mathbb{R}$ such that $g_2 = \lambda_2 \pi_{j_2}, \ldots, g_{m-1} = \lambda_{m-1} \pi_{j_{m-1}}$ and $g_1(e_j) = 0$ for all $j > j_2$ and $g_m(e_j) = 0$ for all $j < j_{m-1}$. Therefore there exist $\mu_1, \ldots, \mu_{j_2}, \eta_{j_{m-1}}, \ldots, \eta_n \in \mathbb{R}$ such that $g_1(x) = \sum\limits_{j=1}^{j_2} \mu_j x_j$ and $g_m(x) = \sum\limits_{j=j_{m-1}}^{n} \eta_j x_j$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}_n$. By a simple calculation it is easy to see that

$$[S] = \begin{bmatrix} \mu_1 & & & & 0 \\ \vdots & & & & \\ \mu_{j_2} & \lambda_{j_2} & & & \\ & & \ddots & & \\ & & & \lambda_{j_{m-1}} & \eta_{j_{m-1}} \\ & & & & \vdots \\ 0 & & & & \eta_n \end{bmatrix} \in \mathbf{M}_{n,m},$$

where λ_{j_k} appears in the $j_k{}^{\text{th}}$ row of $[S]$ for each $2 \leqslant k \leqslant m-1$. To obtain $[T]$ from $[S]$, we have to put zero block columns at the beginning and at the end of $[S]$, and then put some nonzero multiple of any column of $[S]$ in its next column. Therefore (ii) holds with $P = I$. If in the proof we assume that (2) holds, then (ii) holds with P being the backward identity.

Conversely, it is easy to see that if $[T]$ has the form defined in (i) or (ii), then T preserves row-dense vectors. $\square$

Theorem 2.5. *Let $T\colon \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ be a linear map. If $n \geqslant 3$ and T strongly preserves row-dense vectors, then T is invertible.*

Proof. Assume, if possible, that there exists a nonzero matrix $A \in \mathbf{M}_{m,n}$ such that $T(A) = 0$. Since 0 is a row-dense matrix and T strongly preserves row-dense matrices, A is row-dense. Without loss of generality assume that $a = (a_1, \ldots, a_n) \neq 0$ is the first row of A. Now we have the following cases:

Case 1. Let $a_j \neq 0$ for some $1 < j < n$. Take $\lambda, \mu \in \mathbb{R}$ such that $\lambda + a_{j-1} \neq 0 \neq \mu + a_{j+1}$. Define $B = [b_{i,j}] \in \mathbf{M}_{m,n}$ such that all the entries are zero except $b_{1,j-1} = \lambda + a_{j-1}$, $b_{1,j+1} = \mu + a_{j+1}$.

Case 2. Let $a_j = 0$ for all $1 < j < n$. Since A is row-dense, $a_1 \neq 0$ and $a_n = 0$ or $a_1 = 0$ and $a_n \neq 0$. Without loss of generality assume that $a_1 \neq 0$ and $a_n = 0$, and define $B = [b_{i,j}] \in \mathbf{M}_{m,n}$ such that $b_{1,1} = -a_1$, $b_{1,3} = 1$ and zero elsewhere.

In each of the above cases B is not row-dense but $A + B$ is row-dense. Therefore $T(B)$ is not row-dense and $T(A + B) = T(B)$ is row-dense which is a contradiction. $\square$

The following theorem characterizes the linear maps which strongly preserve row-dense vectors.

Theorem 2.6. *Let $T\colon \mathbb{R}_n \to \mathbb{R}_n$ be a linear map. Then T strongly preserves row-dense vectors if and only if there exist $\lambda_1, \ldots, \lambda_n \neq 0$ and $\lambda_0, \lambda_{n+1} \in \mathbb{R}$ such that*

$$[T] = \begin{bmatrix} \lambda_1 & 0 & & & & & 0 \\ \lambda_0 & \lambda_2 & & & & & \\ & & \lambda_3 & & & & \\ & & & \ddots & & & \\ & & & & \lambda_{n-2} & & \\ & & & & & \lambda_{n-1} & \lambda_{n+1} \\ 0 & & & & & 0 & \lambda_n \end{bmatrix} P, \tag{2.3}$$

where P is the identity or the backward identity matrix of order n.

Proof. First assume that $[T]$ has the form (2.3). It is then easy to see that T is invertible and that

$$[T^{-1}] = (PT'P)P$$

where

$$T' = \begin{bmatrix} 1/\lambda_1 & 0 & & & & & 0 \\ -\lambda_0/\lambda_1\lambda_2 & 1/\lambda_2 & & & & & \\ & & 1/\lambda_3 & & & & \\ & & & \ddots & & & \\ & & & & 1/\lambda_{n-2} & & \\ & & & & & 1/\lambda_{n-1} & -\lambda_{n+1}/\lambda_{n-1}\lambda_n \\ 0 & & & & & 0 & 1/\lambda_n \end{bmatrix}.$$

Therefore by Theorem 2.4, T and T^{-1} preserve row-dense vectors and hence T strongly preserves row-dense vectors. Conversely, assume that T strongly preserves row-dense vectors. By Theorem 2.4, $[T]$ has the form (2.2), and by Theorem 2.5, $[T]$ is invertible. Therefore $[T]$ has the form (2.3) and the proof is complete. $\square$

3. Linear preservers of row-dense matrices

In this section we find the structure of linear maps $T\colon \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ that preserve (strongly preserve) row-dense matrices. For $X = [x_{ij}]$, $Y = [y_{ij}] \in \mathbf{M}_{m,n}$, by $\langle X, Y\rangle$ we mean the usual inner product on $\mathbf{M}_{m,n}$, i.e. $\langle X, Y\rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}y_{ij}$ and by $X \circ Y$ we mean the Hadamard product on $\mathbf{M}_{m,n}$, i.e. $X \circ Y = [x_{ij}y_{ij}]$. The matrix $E_{ij} \in \mathbf{M}_{m,n}$ has 1 in position (i, j) and zeros elsewhere.

Lemma 3.1. *Let $T\colon \mathbf{M}_{m,n} \to \mathbb{R}_n$ be a linear map. Then T preserves row-dense matrices if and only if one of the following holds:*

(i) *There exist a row-dense vector $b \in \mathbb{R}_n$ and $F, G \in \mathbf{M}_{m,n}$ such that*

$$T(X) = (\langle X, F\rangle, \ldots, \langle X, F\rangle, \langle X, G\rangle, \ldots, \langle X, G\rangle) \circ b, \quad X \in \mathbf{M}_{m,n}, \tag{3.1}$$

where $\langle X, F\rangle$ and $\langle X, G\rangle$ are repeated j and $n - j$ times, respectively, for some $0 \leqslant j \leqslant n$.

(ii) *There exist $1 \leqslant k \leqslant m$ and $H \in \mathbf{M}_n$ of the form (2.2) such that*

$$T(X) = x_k H, \quad X \in \mathbf{M}_{m,n}, \tag{3.2}$$

where x_k is the k^{th} row of X.

P r o o f. Since the sufficiency of the conditions is easy to see, we only need to prove the necessity of the conditions. Assume that T preserves row-dense matrices. Let $T = (T_1, \ldots, T_n)$, where for every $1 \leqslant j \leqslant n$, $T_j\colon \mathbf{M}_{m,n} \to \mathbb{R}$ is the j^{th} component of T. Then for every $1 \leqslant j \leqslant n$ there exists $F_j \in \mathbf{M}_{m,n}$ such that

$$T_j(X) = \langle X, F_j \rangle, \quad X \in \mathbf{M}_{m,n}.$$

We assume that (i) does not hold and then we show that (ii) holds. Since (i) does not hold, there exist $1 \leqslant i < j < k \leqslant n$ such that $\{T_i, T_j\}$ and $\{T_j, T_k\}$ are linearly independent. Without loss of generality we may assume that $\{T_1, T_2\}$ and $\{T_2, T_3\}$ are linearly independent. Let $\varphi\colon \mathbb{R}_{mn} \to \mathbf{M}_{m,n}$ be the natural isomorphism. Then $\{T_1 \circ \varphi, T_2 \circ \varphi\}$ and $\{T_2 \circ \varphi, T_3 \circ \varphi\}$ are linearly independent. Since $T \circ \varphi$ preserves row-dense vectors, by Lemma 2.2, $T_2 \circ \varphi$ is a multiple of an axes projection and hence there exist $1 \leqslant k \leqslant m$ and $1 \leqslant l \leqslant n$ such that T_2 is a multiple of the projection onto the entry (k, l). Now, we show that $T_1(E_{ij}) = 0$ for $i \neq k$. Assume, if possible, that there exists a pair (p, q) such that $T_1(E_{pq}) \neq 0$ and $p \neq k$. Since $\{T_2, T_3\}$ is linearly independent, there exists a pair $(r, s) \neq (k, l)$ such that $T_3(E_{rs}) \neq 0$. If $p \neq r$, put $X = \lambda E_{pq} + \mu E_{rs}$ for some $\lambda, \mu \in \mathbb{R}$ such that $X \notin \operatorname{Ker}(T_1) \cup \operatorname{Ker}(T_3)$. If $p = r$ and $q \leqslant s$, put $X = \lambda E_{pq} + E_{p(q+1)} + \ldots + E_{p(s-1)} + \mu E_{ps}$ for some $\lambda, \mu \in \mathbb{R}$ such that $X \notin \operatorname{Ker}(T_1) \cup \operatorname{Ker}(T_3)$. The case where $p = r$ and $s < q$ is similar. It is clear that in all these cases X is row-dense but $T(X)$ is not row-dense, which is a contradiction. Hence, $T_1(E_{ij}) = T_2(E_{ij}) = 0$ for $i \neq k$. Similarly we can show that $T_3(E_{ij}) = 0$ for $i \neq k$.

For $n \geqslant 4$ consider $\{T_2, T_3\}$ and $\{T_3, T_4\}$. If $\{T_3, T_4\}$ is linearly dependent, then also $T_4(E_{ij}) = 0$ for $i \neq k$. If $\{T_3, T_4\}$ is linearly independent, with the use Lemma 2.2, we also have that $T_4(E_{ij}) = 0$ for $i \neq k$. We can continue this process and obtain $T_1(E_{ij}) = \ldots = T_n(E_{ij}) = 0$ for $i \neq k$. So there exists $H \in \mathbf{M}_n$ such that

$$T(X) = x_k H, \quad X \in \mathbf{M}_{m,n}, \tag{3.3}$$

where x_k is the k^{th} row of X.

Finally, consider the linear map $S\colon \mathbb{R}_n \to \mathbf{M}_{m,n}$ defined by $S(x) = e_k^t x$. ($S(x)$ is the $m \times n$ matrix with x in the k^{th} row and zeros elsewhere.) Observe that by (3.3), $(T \circ S)(x) = xH$ for all $x \in \mathbb{R}_n$. Since T preserves row-dense matrices, $T \circ S\colon \mathbb{R}_n \to \mathbb{R}_n$ preserves row-dense vectors. By Theorem 2.4, H is of the form (2.2) and that completes the proof. □

Theorem 3.2. *Let $T\colon \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ be a linear map. Then T preserves row-dense matrices if and only if there exist a permutation matrix $Q \in \mathbf{M}_m$, a row-dense* *matrix* $R \in \mathbf{M}_{k,n}$, $F_1, G_1, \ldots, F_k, G_k \in \mathbf{M}_{m,n}$, *and* $H_{k+1}, \ldots, H_m \in \mathbf{M}_n$ *of the form* (2.2) *in Theorem* 2.4 *such that for all* $X \in \mathbf{M}_{m,n}$

$$T(X) = Q\left(\begin{bmatrix} \langle X, F_1\rangle, \ldots, \langle X, F_1\rangle, \langle X, G_1\rangle, \ldots, \langle X, G_1\rangle \\ \vdots \\ \langle X, F_k\rangle, \ldots, \langle X, F_k\rangle, \langle X, G_k\rangle, \ldots, \langle X, G_k\rangle \\ \hline x_{i_{k+1}} H_{k+1} \\ \vdots \\ x_{i_m} H_m \end{bmatrix} \circ \begin{bmatrix} R \\ \text{—} \\ J \end{bmatrix}\right), \tag{3.4}$$

where x_j *is the* j^{th} *row of* X *for every* $1 \leqslant i \leqslant k$, $\langle X, F_i\rangle$ *and* $\langle X, G_i\rangle$ *are repeated* l_i *and* $n - l_i$ *times, respectively, for some* $0 \leqslant l_i \leqslant n$, *and* $J \in \mathbf{M}_{m-k,n}$ *is the all* 1*'s matrix. (Note that* $i_{k+1}, \ldots, i_m$ *are not necessarily distinct.)*

Proof. If (3.4) holds, it is easy to see that T preserves row-dense matrices. Conversely, assume that T preserves row-dense matrices. Let $T = \begin{pmatrix} T_1 \\ \vdots \\ T_m \end{pmatrix}$, where $T_j \colon \mathbf{M}_{m,n} \to \mathbb{R}_n$ is the j^{th} component of T. Since T_j preserves row-dense matrices, by Lemma 3.1, each T_j has the form (3.1) or (3.2). Assume that k components of T have the form (3.1) and $(m-k)$ components have the form (3.2) for some $0 \leqslant k \leqslant m$. So, by using a suitable permutation matrix $Q \in \mathbf{M}_m$ we may assume that $T_1, \ldots, T_k$ have the form (3.1), $T_{k+1}, \ldots, T_m$ have the form (3.2) and $T(X) = Q\begin{pmatrix} T_1(X) \\ \vdots \\ T_m(X) \end{pmatrix}$ for all $X \in \mathbf{M}_{m,n}$. Hence, for every $1 \leqslant j \leqslant k$, there exist $F_j, G_j \in \mathbf{M}_{m,n}$ and a row-dense vector $b_j \in \mathbb{R}_n$ such that for all $X \in \mathbf{M}_{m,n}$, $T_j(X) = (\langle X, F_j\rangle, \ldots, \langle X, F_j\rangle, \langle X, G_j\rangle, \ldots, \langle X, G_j\rangle) \circ b_j$ where $\langle X, F_j\rangle$ and $\langle X, G_j\rangle$ are repeated l_j and $(n - l_j)$ times, respectively, for some $0 \leqslant l_j \leqslant n$; and for every $k+1 \leqslant j \leqslant m$, there exists $H_j \in \mathbf{M}_n$ of the form (2.2) in Theorem 2.4, and $1 \leqslant i_j \leqslant m$, such that for all $X \in \mathbf{M}_{m,n}$, $T_j(X) = x_{i_j} H_j$. Put $R = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix} \in \mathbf{M}_{k,n}$, so that R is a row-dense matrix, and then it is easy to see that (3.4) holds. □

Theorem 3.3. *Let* $T \colon \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ *be a linear map. Then the following statements are equivalent:*

(i) T *strongly preserves row-dense matrices.*

(ii) T *is invertible and* T *preserves row-dense matrices.*

(iii) *There exist a permutation matrix $Q \in \mathbf{M}_m$, and matrices $H_1, \dots, H_m \in \mathbf{M}_n$ of the form (2.3) in Theorem 2.6 such that*

$$T(X) = Q \begin{pmatrix} x_1 H_1 \\ \vdots \\ x_m H_m \end{pmatrix}, \quad X \in \mathbf{M}_{m,n},$$

where x_j is the j^{th} row of X for $1 \leqslant j \leqslant m$.

Proof. (i)→(ii) is a consequence of Theorem 2.5. To prove (ii)→(iii), let $T = \begin{pmatrix} T_1 \\ \vdots \\ T_m \end{pmatrix}$, where $T_j\colon \mathbf{M}_{m,n} \to \mathbb{R}_n$ is the j^{th} component of T. By Lemma 3.1, each T_j has the form (3.1) or (3.2). We show that no T_j has the form (3.1) when T strongly preserves row-dense matrices. Assume, if possible, that k, $1 \leqslant k \leqslant m$, components of T have the form (3.1). Without loss of generality we may assume that $T_1, \dots, T_k$ have the form (3.1) and $T_{k+1}, \dots, T_m$ have the form (3.2). Thus, for every $1 \leqslant j \leqslant k$, there exist $F_j, G_j \in \mathbf{M}_{m,n}$ and a row-dense vector $b_j \in \mathbb{R}_n$ such that

$$T_j(X) = (\langle X, F_j\rangle, \dots, \langle X, F_j\rangle, \langle X, G_j\rangle, \dots, \langle X, G_j\rangle) \circ b_j, \quad X \in \mathbf{M}_{m,n},$$

and for every $k+1 \leqslant j \leqslant m$ there exist $1 \leqslant i_j \leqslant m$ and $H_j \in \mathbf{M}_n$ of the form (2.2) such that

$$T_j(X) = x_{i_j} H_j, \quad X \in \mathbf{M}_{m,n}.$$

Put $V = \{X \in \mathbf{M}_{m,n}\colon x_{i_j} = 0,\ \text{for } k+1 \leqslant j \leqslant m\}$. It is clear that $\dim(V) \geqslant nk$. Now, for every $1 \leqslant j \leqslant k$ consider the linear functionals $f_j, g_j\colon V \to \mathbb{R}$ defined by $f_j(X) = \langle X, F_j\rangle$ and $g_j(X) = \langle X, G_j\rangle$, respectively.

For $1 \leqslant j \leqslant k$, form F'_j (or G'_j) from F_j (or G_j) by replacing rows $i_{k+1}, \dots, i_m$ in F_j (or G_j) by zero rows. Then $W = \operatorname{Span}\{F'_1, G'_1, \dots, F'_k, G'_k\}$ is a subspace of V. Note that $f_j(X) = \langle X, F'_j\rangle$ and $g_j(X) = \langle X, G'_j\rangle$ for all $X \in V$. Since $n \geqslant 3$, $\dim(W) \leqslant 2k < nk \leqslant \dim(V)$. Using the fact that $V = W \oplus W^\perp$, one can see that there exists $0 \neq X \in W^\perp$. So $f_j(X) = g_j(X) = 0$ for every $1 \leqslant j \leqslant m$. On the other hand, since $X \in V$, $T_j(X) = 0$ for every $k+1 \leqslant j \leqslant m$. Therefore $T(X) = 0$ which is a contradiction, and hence for all $1 \leqslant j \leqslant m$, T_j has the form (3.2). Since T is invertible, there is a one-to-one correspondence between $\{i_1, \dots, i_m\}$ and $\{1, \dots, m\}$. So by renaming $H_1, \dots, H_m$ and choosing a suitable permutation matrix $Q \in \mathbf{M}_m$ we obtain $T(X) = Q \begin{pmatrix} x_1 H_1 \\ \vdots \\ x_m H_m \end{pmatrix}$ for all $X \in \mathbf{M}_{m,n}$. The proof of (iii)→(i) is easy to see. □

Let $\mathbb{R}^n = \mathbf{M}_{n,1}$. Now, we consider column-dense vectors and column-dense matrices as follows:

Definition 3.4. A vector $a \in \mathbb{R}^n$ is called a *column-dense vector* if there are no zeros between two nonzero components of a. A matrix $A \in \mathbf{M}_{n,m}$ is called a *column-dense matrix* if every column of A is a column-dense vector.

Let $T\colon \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear map. Define the linear map $S\colon \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ by $S(X) = (T(X^t))^t$. It is easy to see that S (strongly) preserves row-dense matrices if and only if T (strongly) preserves column-dense matrices. Therefore, all previous results are applicable to S. The following theorems characterize the strong linear preservers of column-dense matrices. The linear preservers of column-dense matrices can be presented similarly.

Theorem 3.5. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Then T strongly preserves column-dense vectors if and only if there exist $\lambda_1, \ldots, \lambda_n \neq 0$ and $\lambda_0, \lambda_{n+1} \in \mathbb{R}$ such that*

$$[T] = P \begin{bmatrix} \lambda_1 & \lambda_0 & & & & & 0 \\ 0 & \lambda_2 & & & & & \\ & & \lambda_3 & & & & \\ & & & \ddots & & & \\ & & & & \lambda_{n-2} & & \\ & & & & & \lambda_{n-1} & 0 \\ 0 & & & & & \lambda_{n+1} & \lambda_n \end{bmatrix}, \tag{3.5}$$

where P is the identity or the backward identity matrix of order n.

Theorem 3.6. *Let $T\colon \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear map. Then T strongly preserves column-dense matrices if and only if T is invertible and T preserves column-dense matrices if and only if there exist a permutation matrix $Q \in \mathbf{M}_m$, and matrices $F_1, \ldots, F_m \in \mathbf{M}_n$ of the form* (3.5) *in Theorem* 3.5*, such that*

$$T(X) = (F_1 x_1 \ldots F_m x_m)Q, \quad X \in \mathbf{M}_{n,m},$$

where x_j is the j^{th} column of X.

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